NON-LOCALITY OF EQUIVARIANT STAR PRODUCTS ON $\mathbf{T}^*(\mathbb{RP}^n)$

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Abstract. Lecomte and Ovsienko constructed $SL_{n+1}(\mathbb{R})$-equivariant quantization maps $\mathcal{Q}_\lambda$ for symbols of differential operators on λ-densities on $\mathbb{RP}^n$.

We derive some formulas for the associated graded equivariant star products $\star_\lambda$ on the symbol algebra $\mathrm{Pol}(T^*\mathbb{RP}^n)$. These give some measure of the failure of locality.

Our main result expresses (for n odd) the coefficients $C_p(\cdot,\cdot)$ of $\star_\lambda$ when $\lambda=\frac{1}{2}$ in terms of some new $SL_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operators $Z_p(\cdot,\cdot)$ on $T^*\mathbb{CP}^n$ and the operators $(E+\frac{n}{2}\pm s)^{-1}$ where E is the fiberwise Euler vector field and $s\in\{1,2,\cdots,[\frac{p}{2}]\}$.

1. Introduction

Lecomte and Ovsienko ([L-O]) constructed $SL_{n+1}(\mathbb{R})$-equivariant quantization maps $\mathcal{Q}_\lambda$ for symbols of differential operators on λ-densities on $\mathbb{RP}^n$.

We derive some formulas for the associated graded equivariant star products $\phi\star_\lambda\psi = \phi\psi+\sum_{p=1}^{\infty}C_p^\lambda(\phi,\psi)t^p$ on the symbol algebra $\mathrm{Pol}_\infty(T^*\mathbb{RP}^n)$. The star products $\star_\lambda$ is "algebraic" in that (Proposition 3.1) it restricts to the subalgebra $\mathcal{R}$ generated by the momentum functions μ^x, $x\in\mathfrak{sl}_{n+1}(\mathbb{R})$.

We compute some special values of $\phi\star_\lambda\psi$ in Proposition 4.1. We conclude in Corollary 4.2 that $C_p^\lambda(\cdot,\cdot)$ fails to be bidifferential, except if $\lambda=\frac{1}{2}$ and $p=1$. The reason is that $C_p^\lambda(\cdot,\cdot)$ involves operators of the form $(E+r)^{-1}$ where E is the fiberwise Euler vector field on $T^*\mathbb{RP}^n$ and r is a positive number.

In our main result (Theorem 5.1), we write, for n odd, the coefficients $C_p^\lambda(\cdot,\cdot)$ when $\lambda=\frac{1}{2}$ in terms of some new $SL_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operators $Z_p(\cdot,\cdot)$ on $\mathbb{CP}^n$ and the operators $(E+\frac{n}{2}\pm s)^{-1}$ where $s\in\{1,2,\cdots,[\frac{p}{2}]\}$. Our proofs in §4-§5 are applications of the formulas in [L-O, §4.5] for $\mathcal{Q}_\lambda$.

The operator $Z_p(\cdot,\cdot)$ $(p\geq 2)$ is quite subtle as it has total homogeneous degree $-p$. It is not the pth power of the Poisson tensor (with respect to some coordinates) because we can show that the total order of $Z_p(\cdot,\cdot)$ is too large. It would be very interesting to find a way to construct Z_p using the method of Levasseur and Stafford ([L-S]).

I thank Christian Duval and Valentin Ovsienko for several interesting discussions.

2. The Lecomte-Ovsienko quantization maps

In [L-O], Lecomte and Ovsienko constructed, for each $\lambda\in\mathbb{C}$, an $SL_{n+1}(\mathbb{R})$-equivariant (complex linear) quantization map $\mathcal{Q}_\lambda$ from $\mathcal{A}=\mathrm{Pol}_\infty(T^*\mathbb{RP}^n)$ to $\mathcal{B}^\lambda=\mathfrak{D}^\lambda_\infty(\mathbb{RP}^n)$. Here $\mathcal{A}=\oplus_{d=0}^{\infty}\mathcal{A}^d$ is the graded Poisson algebra of smooth complex-valued functions on $T^*\mathbb{RP}^n$ which are polynomial along the cotangent fibers, and $\mathcal{B}^\lambda=\cup_{d=0}^{\infty}\mathcal{B}^\lambda_d$ is the filtered algebra of smooth (linear) differential operators on λ-densities on $\mathbb{RP}^n$. Then

$\mathcal{Q}_\lambda$ is a quantization map in the sense that $\mathcal{Q}_\lambda$ is a vector space isomorphism and ϕ is the principal symbol of $\mathcal{Q}_\lambda(\phi)$ if $\phi \in \mathcal{A}^d$.

The natural action of $SL_{n+1}(\mathbb{R})$ on $\mathbb{RP}^n$ lifts canonically to a Hamiltonian action on $T^*\mathbb{RP}^n$ with moment map $\mu : T^*\mathbb{RP}^n \to \mathfrak{sl}_{n+1}(\mathbb{R})^*$. The density line bundle on $\mathbb{RP}^n$ is homogeneous for $SL_{n+1}(\mathbb{R})$. This geometry produces natural (complex linear) representations of $SL_{n+1}(\mathbb{R})$ on $\mathcal{A}$ and $\mathcal{B}^\lambda$; $\mathcal{Q}_\lambda$ is equivariant for these representations.

The procedure of Lecomte and Ovsienko was to construct ([L-O, Thm. 4.1]) an $\mathfrak{sl}_{n+1}(\mathbb{R})$-equivariant quantization map $\mathcal{Q}_\lambda$ from $\mathrm{Pol}_\infty(T^*\mathbb{R}^n)$ to $\mathfrak{D}^\lambda_\infty(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the big cell in $\mathbb{RP}^n$. They show their map is unique. Then $\mathcal{Q}_\lambda$ restricts to a quantization map from $\mathcal{A}$ to $\mathcal{B}^\lambda$ ([L-O, Cor. 8.1]).

We can represent points in $\mathbb{RP}^n$ in homogeneous coordinates $[u_0, \dots, u_n]$. Then $u_1, \dots, u_n$ are linear coordinates on the big cell $\mathbb{R}^n$ defined by $u_0 = 1$. These, together with the conjugate momenta $\xi_1, \dots, \xi_n$, give Darboux coordinates on $T^*\mathbb{R}^n$.

For any vector field η on $\mathbb{RP}^n$, let $\mu_\eta \in \mathcal{A}^1$ be its principal symbol and let η_λ be its Lie derivative acting on λ-densities so that $\eta_\lambda \in \mathcal{B}^\lambda_1$. Then $\mathcal{Q}_\lambda(\mu_\eta) = \eta_\lambda$; this follows by [L-O, §4.3].

The quantization map $\mathcal{Q}_\lambda$ defines a star product; see [L-O, §8.2]. For $\phi, \psi \in \mathcal{A}$, we put $\phi \star_\lambda \psi = \mathcal{Q}^{-1}_{\lambda;t}(\mathcal{Q}_{\lambda;t}(\phi)\mathcal{Q}_{\lambda;t}(\psi))$ where $\mathcal{Q}_{\lambda;t}$ is the linear map $\mathcal{A} \to \mathcal{B}^\lambda[t]$ such that $\mathcal{Q}_{\lambda;t}(\phi) = t^d \mathcal{Q}_\lambda(\phi)$ if $\phi \in \mathcal{A}^d$. Then $\star_\lambda$ makes $\mathcal{A}[t]$ into an associative algebra over $\mathbb{C}[t]$. This satisfies

$$\phi \star_\lambda \psi = \textstyle\sum_{p=0}^\infty C^\lambda_p(\phi, \psi) t^p \tag{2.1}$$

where $C^\lambda_0(\phi, \psi) = \phi\psi$ and $C^\lambda_1(\phi, \psi) - C^\lambda_1(\psi, \phi) = \{\phi, \psi\}$. Also $C^\lambda_p(\phi, \psi) \in \mathcal{A}^{j+k-p}$ if $\phi \in \mathcal{A}^j$ and $\psi \in \mathcal{A}^k$. So $\star_\lambda$ is a graded star product on $\mathcal{A}$.

We say that $\star_\lambda$ has *parity* if $C^\lambda_p(\phi, \psi) = (-1)^p C^\lambda_p(\psi, \phi)$; then $C^\lambda_1(\phi, \psi) = \frac{1}{2}\{\phi, \psi\}$.

Lemma 2.1. *$\star_\lambda$ has parity iff $\lambda = \frac{1}{2}$.*

Proof. Let $\beta : \mathcal{B}^\lambda \to \mathcal{B}^{1-\lambda}$ be the canonical algebra anti-isomorphism and let $\alpha : \mathcal{A} \to \mathcal{A}$ be the Poisson algebra anti-involution defined by $\phi^\alpha = (-1)^d \phi$ if $\phi \in \mathcal{A}^d$. Then $\mathcal{Q}_\lambda(\phi^\alpha)^\beta = \mathcal{Q}_{1-\lambda}(\phi)$ by [L-O, Lem. 6.5]. This implies $C^\lambda_p(\phi, \psi) = (-1)^p C^{1-\lambda}_p(\psi, \phi)$. So we have parity if $\lambda = \frac{1}{2}$. Otherwise parity is violated, already for C^λ_1. Indeed, if $\phi \in \mathcal{A}^0$ and $\mu \in \mathcal{A}^1$, then $\phi \star_\lambda \mu = \phi\mu + \lambda\{\phi, \mu\}t$, and so $C^\lambda_1(\phi, \mu) = \lambda\{\phi, \mu\}$ while $C^\lambda_1(\mu, \phi) = -C^{1-\lambda}_1(\phi, \mu) = (\lambda - 1)\{\phi, \mu\}$. ∎

3. Algebraicity of $\star_\lambda$

Each $x \in \mathfrak{sl}_{n+1}(\mathbb{R})$ defines a vector field η^x on $T^*\mathbb{RP}^n$. The principal symbols $\mu^x = \mu_{\eta^x}$ are the momentum functions for $SL_{n+1}(\mathbb{R})$. The $SL_{n+1}(\mathbb{R})$-equivariance of $\mathcal{Q}_\lambda$ is equivalent to $\mathfrak{sl}_{n+1}(\mathbb{R})$-equivariance, i.e., $\mathcal{Q}_\lambda(\{\mu^x, \phi\}) = [\eta^x_\lambda, \mathcal{Q}_\lambda(\phi)]$. Then $\mathcal{Q}_\lambda$ is $\mathfrak{sl}_{n+1}(\mathbb{C})$-equivariant, where we define μ^x and η^x_λ for $x \in \mathfrak{sl}_{n+1}(\mathbb{C})$ by $\mu^{x+iy} = \mu^x + i\mu^y$ and so on.

The algebra $R(T^*\mathbb{CP}^n)$ of regular functions (in the sense of algebraic geometry) on (the quasi-projective complex algebraic variety) $T^*\mathbb{CP}^n$ identifies, by restriction, with a subalgebra $\mathcal{R}$ of $\mathcal{A}$. Similarly the algebra of $\mathfrak{D}^\lambda(\mathbb{CP}^n)$ of twisted algebraic (linear)

differential operators for the formal λth power of the canonical bundle $\mathcal{K}$ identifies with a subalgebra $\mathcal{D}^\lambda$ of $\mathcal{B}^\lambda$.

Then $\mathcal{R}$ is generated by the momentum functions μ^x, $\mathcal{D}^\lambda$ is generated by the operators η^x_λ, and $\operatorname{gr}\mathcal{D}^\lambda = \mathcal{R}$. These statements follow, for instance, by [Bo-Br, Lem. 1.4 and Thm. 5.6], since the proofs of the relevant results there generalize immediately to the twisted case. We get natural identifications $\mathcal{R} = \mathcal{S}/I$ and $\mathcal{D}^\lambda = \mathcal{U}(\mathfrak{g})/J$ where I is graded Poisson ideal in the symmetric algebra $\mathcal{S} = S(\mathfrak{sl}_{n+1}(\mathbb{C}))$, J is a two-sided ideal in the enveloping algebra $\mathcal{U} = \mathcal{U}(\mathfrak{sl}_{n+1}(\mathbb{C}))$, and $\operatorname{gr} J = I$.

Notice $\mathcal{R}$ carries a natural representation of $SL_{n+1}(\mathbb{C})$, which then extends the $SL_{n+1}(\mathbb{R})$-symmetry it inherits from $\mathcal{A}$.

Proposition 3.1. *For every λ, $\star_\lambda$ restricts to a graded G-equivariant star product on the momentum algebra $\mathcal{R}$.*

Proof. It suffices to check that $\mathcal{Q}_\lambda$ maps $\mathcal{R}$ onto $\mathcal{D}^\lambda$ (which is stated for $\lambda = 0$ in [L-O, §1.5, Remark (c)]). This follows easily in any number of ways. For instance, the formula for $\mathcal{Q}_\lambda$ in [L-O, (4.15)] implies $\mathcal{Q}_\lambda(\xi_1^{a_1}\cdots\xi_n^{a_n}) = \frac{\partial^{a_1}}{\partial u_1^{a_1}}\cdots\frac{\partial^{a_n}}{\partial u_n^{a_n}}$. But $\{\xi_n^d\}_{d=0}^\infty$ and $\{\frac{\partial^d}{\partial u_n^d}\}_{d=0}^\infty$ are complete sets of lowest weight vectors in $\mathcal{R}$ and $\mathcal{D}^\lambda$. $\square$

Remark 3.2. The restriction of $\star_\lambda$ to $\mathcal{R}$ has parity iff (i) $\lambda = \frac{1}{2}$ or (ii) $n = 1$; see [A-B2, §3]. Notice that (ii) does not contradict the proof of Lemma 2.1, as $\mathcal{R}^0 = \mathbb{C}$.

4. Some special values of $\phi \star_\lambda \psi$

$\mathrm{Pol}_\infty(T^*\mathbb{R}^n)$ is the tensor product of two maximal Poisson commutative subalgebras, namely the algebra $\mathbb{C}_\infty[u] = \mathbb{C}_\infty[u_1, \ldots, u_n]$ of smooth functions on the big cell $\mathbb{R}^n$ and and the polynomial algebra $\mathbb{C}[\xi] = \mathbb{C}[\xi_1, \ldots, \xi_n]$. Let E be the fiberwise Euler vector field $\sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}$. Set $D = \sum_{i=1}^n \frac{\partial^2}{\partial u_i \partial \xi_i}$.

Proposition 4.1. *If $\phi \in \mathbb{C}_\infty[u]$ and $\psi \in \mathbb{C}[\xi]$ then $\phi \star_\lambda \psi = \mathbf{g}_\lambda(\phi\psi)$ where*

$$\mathbf{g}_\lambda = 1 + \sum_{d=1}^\infty g_{\lambda;d} D^d t^d \qquad \textit{and} \qquad g_{\lambda;d} = \frac{1}{d!}\prod_{j=0}^{d-1} \frac{-E - j - \lambda(n+1)}{2E + j + n + 1} \tag{4.1}$$

Proof. Let $\mathcal{Q}_{norm} : \mathrm{Pol}_\infty(T^*\mathbb{R}^n) \to \mathfrak{D}_\infty(\mathbb{R}^n)$ be the normal ordering quantization map. The construction of $\mathcal{Q}_\lambda$ in [L-O] gives $\mathcal{Q}_\lambda = \mathcal{Q}_{norm}\mathbf{h}_\lambda$ where $\mathbf{h}_\lambda = 1 + \sum_{d=1}^\infty h_{\lambda;d} D^d t^d$ and $h_{\lambda;d}$ are certain operators. Here $\mathfrak{D}_\infty(\mathbb{R}^n)$ identifies with $\mathfrak{D}^\lambda_\infty(\mathbb{R}^n)$ in the usual way.

In [D-L-O, Th. 4.1] they give a very nice formula for the $h_{\lambda;d}$ when $\lambda = \frac{1}{2}$. Going back to [L-O, (4.15)], we get a similar formula for all λ. We find

$$h_{\lambda;d} = \frac{1}{d!}\prod_{j=0}^{d-1} \frac{E + j + \lambda(n+1)}{2E + j + n + d} \tag{4.2}$$

Thus for $\phi, \psi \in \mathrm{Pol}_\infty(T^*\mathbb{R}^n)$ we have

$$\phi \star_\lambda \psi = \mathbf{g}_\lambda(\mathbf{h}_\lambda(\phi) \# \mathbf{h}_\lambda(\psi)) \tag{4.3}$$

where # denotes the graded star product defined by $\mathcal{Q}_{norm}$ and $\mathbf{g}_\lambda = \mathbf{h}_\lambda^{-1}$. We find, directly from (4.2) or using [L-O, (4.10)], that $\mathbf{g}_\lambda$ is given by (4.1).

We know $\phi\#\psi = \sum_{p=0}^{\infty} N_p(\phi,\psi)t^p$ where $N_k(\phi,\psi) = \frac{1}{k!}\sum_{\alpha\in\{1,\ldots,n\}^k} \frac{\partial^k\phi}{\partial\xi_\alpha}\frac{\partial^k\psi}{\partial u_\alpha}$. Now, for $\phi\in\mathbb{C}_\infty[u]$ and $\psi\in\mathbb{C}[\xi]$, (4.3) gives $\phi\star_\lambda\psi = \mathbf{g}_\lambda(\phi\psi)$. $\square$

Corollary 4.2. *None of the operators C_p^λ ($p\ge 1$, $\lambda\in\mathbb{C}$) is bidifferential on $T^*\mathbb{R}^n$, with one exception: $2C_1^{\frac12}$ is the Poisson bracket.*

Proof. We just showed that $C_p^\lambda(\phi,\psi) = g_{\lambda;p}D^p(\phi\psi)$ if $\phi\in\mathbb{C}_\infty[u]$ and $\psi\in\mathbb{C}[\xi]$. This implies, if C_p^λ is bidifferential, that $g_{\lambda;p}$ is a differential operator on $T^*\mathbb{R}^n$. Looking at our expression for $g_{\lambda;p}$, we deduce $E+j+\lambda(n+1) = E+\frac{j}{2}+\frac12(n+1)$ for $j=0,\ldots,p-1$. But this forces $p=1$ and $\lambda=\frac12$. By parity, $C_1^{\frac12} = \frac12\{\cdot,\cdot\}$. $\square$

The corollary contradicts the claim in [L-O, §8.2]. They no doubt meant that for each pair j,k, the restricted map $C_p^\lambda : \mathcal{A}^j\times\mathcal{A}^k\to\mathcal{A}^{j+k-p}$ is given by some bidifferential operator.

5. Coefficients C_p^λ for $\lambda=\frac12$

In this section, we set $\lambda=\frac12$ and suppress the corresponding super(sub)scripts. We put $E' = E+\frac{n}{2}$ where E is the fiberwise Euler vector field on $T^*\mathbb{RP}^n$. See [A-B3] for an interpretation of the shift $\frac{n}{2}$. Let $[m]$ be the greatest integer not exceeding m.

We put $T_p = \prod_{i=1}^{[\frac{p}{2}]}(E'+i)$ and $S_p = \prod_{i=1}^{[\frac{p}{2}]}(E'-i)$. These are both invertible on $\mathcal{A}$ if n is odd. Our main result is

Theorem 5.1. *Assume n is odd and let $p\ge1$. Then C_p has the form*

$$C_p(\phi,\psi) = \frac{1}{T_p}Z_p\left(\frac{1}{S_p}\phi,\frac{1}{S_p}\psi\right),\qquad \phi,\psi\in\mathcal{A} \tag{5.1}$$

where Z_p is an $SL_{n+1}(\mathbb{R})$-invariant bidifferential operator on $T^*\mathbb{RP}^n$.

Z_p is uniquely determined by (5.1), even if we just take $\phi,\psi\in\mathcal{R}$. Thus $\star$ is uniquely determined by its restriction to $\mathcal{R}$, once we know that $(\phi,\psi)\mapsto T_pC_p(S_p\phi,S_p\psi)$ is bidifferential.

Finally, Z_p, like E', extends uniquely to an $SL_{n+1}(\mathbb{C})$-invariant algebraic bidifferential operator on $T^*\mathbb{CP}^n$.

Proof. We return to the proof of Proposition 4.1. Let $\mathbf{g}_d = g_dD^d$ and $\mathbf{h}_d = h_dD^d$, with $\mathbf{g}_0=\mathbf{h}_0=1$. Writing out (4.3) termwise, we get, for $p\ge1$,

$$C_p(\phi,\psi) = \textstyle\sum_{i+j+k+m=p}\mathbf{g}_mN_k(\mathbf{h}_i\phi,\mathbf{h}_j\psi) \tag{5.2}$$

More succinctly, $C_p = \sum_{i+j+k+m=p}\mathbf{g}_mN_k(\mathbf{h}_i\otimes\mathbf{h}_j)$.

For $\lambda=\frac12$, the formula (4.2) simplifies in that $[\frac{d+1}{2}]$ factors cancel out. Then $h_d = U_dV_d^{-1}$ where $U_d = \frac{1}{2^dd!}\prod_{i=1}^{[\frac{d}{2}]}(E'+i-\frac12)$ and $V_d = \prod_{i=[\frac{d+1}{2}]}^{d-1}(E'+i)$. Then $\mathbf{h}_d = U_dV_d^{-1}D^d = U_dD^dS_d^{-1}$. This is a formal relation, valid for n odd since then S_d is invertible. Similarly, (4.1) gives $\mathbf{g}_d = T_d^{-1}F_dD^d$ where $F_d = \frac{1}{2^dd!}\prod_{i=[\frac{d+1}{2}]}^{d-1}(-E'-i-\frac12)$. We put $U_0=F_0=1$.

We put $Z_p(\phi,\psi)=T_pC_p(S_p\phi,S_p\psi)$. Let $T_{p;j}=T_pT_j^{-1}$ and $S_{p;j}=S_j^{-1}S_p$. Now (5.2) gives $Z_p=\sum_{i+j+k+m=p}Z^{mkij}$ where

$$Z^{mkij}=T_{p;m}\,F_mD^mN_k\left(U_iD^iS_{p;i}\otimes U_jD^jS_{p;j}\right) \tag{5.3}$$

Each Z^{mkij}, and so also their sum Z_p, is a bidifferential operator on $T^*\mathbb{R}^n$ with polynomial coefficients. I.e., Z_p lies in $\mathcal{E}\otimes_{\mathcal{P}}\mathcal{E}$ where $\mathcal{E}=\mathbb{C}[u_i,\xi_j,\frac{\partial}{\partial u_k},\frac{\partial}{\partial \xi_l}]$ and $\mathcal{P}=\mathbb{C}[u_i,\xi_j]$.

Now Z_p is invariant under $\mathfrak{sl}_{n+1}(\mathbb{R})$; this is clear since T_p, C_p and S_p are all invariant. It follows by projective geometry (as in [L-O, §8.1]) that Z_p extends uniquely to a global $SL_{n+1}(\mathbb{R})$-invariant bidifferential operator on $T^*\mathbb{RP}^n$.

We have $\{C_p(\phi,\psi)\,|\,\phi,\psi\in\mathcal{R}\}\to\{Z_p(\phi,\psi)\,|\,\phi,\psi\in\mathcal{R}\}\to\{Z_p(\phi,\psi)\,|\,\phi,\psi\in\mathcal{A}\}\to\{C_p(\phi,\psi)\,|\,\phi,\psi\in\mathcal{A}\}$ where the arrows indicate that one set of values completely determines the next set. The middle arrow follows because any bidifferential operator on $T^*\mathbb{RP}^n$ is completely determined by its values on $\mathcal{R}$ ([B, Lemma 5.1]).

Clearly Z_p extends naturally (and uniquely) to an algebraic differential operator $\widetilde{Z}_p$ on $T^*\mathbb{C}^n$; this amounts to replacing our Darboux coordinates u_i,ξ_j by their holomorphic counterparts z_i,ζ_j. Then $\widetilde{Z}_p$ is $\mathfrak{sl}_{n+1}(\mathbb{C})$-invariant and (by projective geometry again) extends to $T^*\mathbb{CP}^n$. $\square$

Notice that this proof gives an explicit formula (in the coordinates u_i,ξ_j) for Z_p.

Remarks 5.2. (i) Suppose n is even. Then this proof still shows that the formula $Z_p(\phi,\psi)=T_pC_p(S_p\phi,S_p\psi)$ defines an operator Z_p in $\mathcal{E}\otimes_{\mathcal{P}}\mathcal{E}$. Then (5.1) is valid as long as ϕ and ψ lie in $\mathcal{A}^*=\oplus_{d=[\frac{p}{2}]-\frac{n}{2}+1}^{\infty}\mathcal{A}^d$. We can show that all the other results in Theorem 5.1 are still true, so that (5.1) determines Z_p uniquely even for $\phi,\psi\in\mathcal{R}\cap\mathcal{A}^*$, Z_p is an $SL_{n+1}(\mathbb{R})$-invariant bidifferential operator on $T^*\mathbb{RP}^n$, etc.

(ii) The maps $\mathcal{Q}_{norm}$ and $\mathbf{h}_\lambda$ are equivariant with respect to only a parabolic subgroup P of $SL_{n+1}(\mathbb{R})$, even though their product $\mathcal{Q}_\lambda=\mathcal{Q}_{norm}\mathbf{h}_\lambda$ is equivariant for $SL_{n+1}(\mathbb{R})$. Here P is the subgroup of the affine transformations of $\mathbb{R}^n$ (i.e., the one which fixes the subspace $(u_0=0)$ in $\mathbb{RP}^n$). Our formula (5.1) is manifestly equivariant for $SL_{n+1}(\mathbb{R})$.

6. Operators $C_p^\lambda(\phi,\cdot)$ for $\lambda=\frac{1}{2}$

Next we recover part of the results found for $\mathfrak{g}=\mathfrak{sl}_{n+1}(\mathbb{C})$ in [A-B1, Prop. 4.2.3] and [A-B2, Thm. 6.3 and Cor. 8.2].

Corollary 6.1. *Let $n\geq 1$. For any momentum function μ^x, $x\in\mathfrak{sl}_{n+1}(\mathbb{C})$, we have*

$$C_2(\mu^x,\psi)=\frac{1}{E'(E'+1)}L^x(\psi),\qquad \psi\in\mathcal{A} \tag{6.1}$$

where L^x is an order 4 differential operator on $T^*\mathbb{RP}^n$.

Neither E' nor $E'+1$ left divides L^x $(x\neq 0)$ over T^*U for any open set U in $\mathbb{RP}^n$. Hence $C_2(\mu^x,\cdot)$ is not a differential operator on T^*U.

Finally, L^x extends uniquely to an algebraic differential operator on $T^*\mathbb{CP}^n$.

Proof. Suppose n is odd. For $\psi\in\mathcal{A}$, (5.1) gives

$$C_2(\mu^x,\psi)=\tfrac{1}{E'+1}Z_2\left(\tfrac{1}{E'-1}\mu^x,\tfrac{1}{E'-1}\psi\right)=\tfrac{2}{nE'(E'+1)}Z_2(\mu^x,\psi) \tag{6.2}$$

The last equality follows because the operator $Z_2(\mu^x,\cdot)$ is graded of degree -1.

For n even, (6.2) is still true on account of Remark 5.2(i), except in the case where $n = 2$ and $\psi \notin \oplus_{d=1}^{\infty} \mathcal{A}^d$. But if $\psi \in \mathcal{A}^0$ then both $C_2(\mu^x, \psi)$ and $Z_2(\mu^x, \psi)$ vanish for degree reasons and so the first and third expressions in (6.2) are still equal.

This proves (6.1), for all n, where $L^x = \frac{2}{n} Z_2(\mu^x, \cdot)$. Then L^x extends to an algebraic differential operator on $T^*\mathbb{CP}^n$; this follows since both Z_2 and μ^x so extend.

The L^x, for $x \neq 0$, all have the same order. This follows because the L^x, like the μ^x, transform in the adjoint representation of $SL_{n+1}(\mathbb{C})$. We can choose $\mu^x = \xi_m$ (the choice of $m \in \{1, \dots, n\}$ is arbitrary). Let $L^{(m)}$ be the corresponding operator L^x. Using (5.2) we find after some calculation

$$C_2(\cdot, \xi_m) = -\frac{1}{16} \frac{1}{E'(E'+1)} \xi_m D^2 + \frac{1}{8} \frac{1}{E'+1} \frac{\partial}{\partial u_m} D \tag{6.3}$$

So $L^{(m)} = -\frac{1}{16}(\xi_m D - 2E' \frac{\partial}{\partial u_m}) D$. Clearly $L^{(m)}$ has order 4. Using principal symbols, we see that $L^{(m)}$ has no left factors of the form $E' + c$ if $n \geq 2$. For $n = 1$, (6.3) gives $L^{(m)} = \frac{1}{16}(E' + \frac{1}{2}) \frac{\partial^3}{\partial u_1^2 \partial \xi_1}$, and so the only such factor is $E' + \frac{1}{2}$. □

Corollary 6.2. *Assume n is odd and let $p \geq 1$. If $\phi \in \mathcal{A}^d$ then*

$$C_p(\phi, \cdot) = \frac{1}{\prod_{i=1}^{[\frac{p}{2}]} (E' + i)(E' - i + p - d)} L_p^{\phi} \tag{6.4}$$

where L_p^{ϕ} is a differential operator on $T^*\mathbb{RP}^n$. If $\phi \in \mathcal{R}$, then L_p^{ϕ} is an algebraic differential operator on $T^*\mathbb{CP}^n$.

Proof. This follows because $L_p^{\phi} = Z_p(S_p^{-1}\phi, \cdot)$. □

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